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THEORY OF LIQUID SLOSHING IN COMPARTMENTED CYLINDRICAL TANKS DUE TO BENDING EXCITATION

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ABSTRACT

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Propellant sloshing may be established by the bending of a space vehicle which exhibits low structural frequencies. Such propellant oscillations are of importance, for there is the possibility of extreme amplitudes if the excitation (bending) frequency is in the neighborhood of one of the natural frequencies of the fuel. Forces and moments exerted by the oscillating propellant on the tank are determined due to forced bending excitation for a liquid in a circular cylindrical ring sector tank with a free fluid surface. Special cases such as the tank with sector and circular cross section are obtained by limit consideration. As is expected, force and moment of the liquid increase sharply at the resonant frequency of the propellant. The total force and moment are generally less for a given maximum bending amplitude than for a translational motion of the same magnitude. The maximum dynamic effects occur when the free fluid surface is located around the point of maximum bending displacement.

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DYNAMICS ANALYSIS BRANCH
FLUTTER AND VIBRATIONS SECTION
AEROBALLISTICS DIVISION

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LIST OF SYMBOLS

Symbol	Definition
$r,\ \varphi,\ z$	Cylindrical coordinates
t	Time
$\emptyset = \emptyset_O + \psi$	Velocity potential
ψ	Disturbance potential
$\emptyset_O$	Potential of liquid without free fluid surface
$\tilde{\zeta}$	Mass density of liquid
ρ	Pressure of liquid
a	Radius of outer tankwell
b	Radius of inner tank wall
$k = a/b$	Diameter ratio
h	Liquid height
g	Longitudinal acceleration (in z - direction)
ω_{mn}	Eigen frequencies of liquid
Ω	Forced circular frequency
$y = \Omega/\omega_{mn}$	Frequency ratio
$x_O(z)$	Amplitude of tank excitation in x - direction
$y_O(z)$	Amplitude of tank excitation in y - direction
F	Fluid force
M	Fluid moment
$\bar{z}$	Force fluid surface displacement, measured from the undisturbed position
$u_r,\ u_\varphi,\ w$	Flow velocity

LIST OF SYMBOLS (CONT'D)

Symbol	Definition
$J_{\frac{m}{2\alpha}}$, $Y_{\frac{m}{2\alpha}}$	Bessel functions of order $m/2\alpha$ of first and second kind
ξ_{mn}	Roots of $\Delta m/2\alpha(\xi) = 0$ (see text)
ϵ_{mn}	Roots of $J'_{\frac{m}{2\alpha}}(\epsilon) = 0$
$2\pi\alpha \equiv \tilde{\alpha}$	Container vertex angle

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SUMMARY

Propellant sloshing may be established by the bending of a space vehicle which exhibits low structural frequencies. Such propellant oscillations are of importance, for there is the possibility of extreme amplitudes if the excitation (bending) frequency is in the neighborhood of one of the natural frequencies of the fuel. Forces and moments exerted by the oscillating propellant on the tank are determined due to forced bending excitation for a liquid in a circular cylindrical ring sector tank with a free fluid surface. Special cases such as the tank with sector and circular cross section are obtained by limit consideration. As is expected, force and moment of the liquid increase sharply at the resonant frequency of the propellant. The total force and moment are generally less for a given maximum bending amplitude than for a translational motion of the same magnitude. The maximum dynamic effects occur when the free fluid surface is located around the point of maximum bending displacement.

I. INTRODUCTION

The constantly increasing size of space vehicles introduces more new problems in modern space technology. As vehicles lengthen, their fundamental bending frequencies become lower; as their diameters become larger, the natural frequency of the propellant becomes lower. These trends restrict the choice of the control frequency value. Considerable and acute problems result from this close grouping of frequencies because of the interaction of structure, control and propellant sloshing.

Performance considerations make necessary a design exhibiting low structural frequencies. Therefore, the effect of elastic vibrations of the vehicle structure upon the propellant sloshing and the control system becomes more critical because of their low frequencies and the overall low damping.

Although the sloshing frequencies are even closer to the control frequency, the effect of propellant sloshing upon stability can be handled more easily than the effect of the elastic structure provided the phases are chosen properly. Of the great many problems involved here, the one of the propellant sloshing due to bending vibrations will be treated.

In the tanks of a missile or space vehicle, sloshing will seriously affect the performance and stability of the vehicle, in some cases, even causing total flight failure. Since more than 90% of the total weight of a vehicle at launch is liquid, propellant sloshing represents an area which needs special attention even in the preliminary design stage. The tendency in modern space technology is toward a continuous increase in size of space vehicles making investigations of this kind mandatory. Therefore, for a realistic dynamic stability and control analysis, even the effect of the oscillating propellant due to bending vibrations of the structure has to be considered.

For space vehicles, which exhibit low propellant and bending frequencies, subdivision of tanks might be of importance to reduce the vibrating sloshing masses and increase the Eigen frequency of the propellant, thus moving it further away from the control frequency. This indicates that the sloshing frequency is becoming closer to the structural frequencies. In the following, therefore, the liquid oscillations in a cylindrical tank with circular ring sector cross section will be treated with respect to bending oscillations. From the results, one can obtain by limit considerations the solutions of the most important cases.

II. FORCED BENDING OSCILLATIONS

The flow field of the liquid with a free fluid surface in a circular ring sector tank with a flat bottom, due to forced bending oscillations of the tank, can be obtained from the solution of the Laplace equation $\Delta \phi = 0$ and the appropriate linearized boundary conditions. The cross section of the tank was considered to be always of identical shape. This will certainly be enhanced by the sector walls. Furthermore, the undisturbed free fluid surface is assumed to stay in the same plane.

The bending of the walls of a tank of an elastic space vehicle can be described as a super-position of translational, rotational, and bending oscillations with a clamped-in tank bottom. This is justified since the theory is linearized.

The response of the liquid due to translational and pitching excitation has been treated previously [1]. The problem left to be solved is the response of the propellant due to bending excitation with a clamped-in tank bottom. The boundary conditions at the tank walls as well as at the free fluid surface (Figure 1) are for arbitrary wall oscillations and are given in linearized form as:

$$\frac{\partial \emptyset}{\partial r} = \begin{Bmatrix} i\Omega\, x_o\,(z)\, e^{i\Omega t} \cos\varphi \\ i\Omega\, y_o\,(z)\, e^{i\Omega t} \sin\varphi \end{Bmatrix} \qquad \text{at the tank wall } r = a, b \qquad (2.1)$$

$$\frac{\partial \emptyset}{\partial z} = 0 \qquad \text{at the tank bottom } z = -h \qquad (2.2)$$

$$\frac{1}{r}\frac{\partial \emptyset}{\partial \varphi} = \begin{Bmatrix} 0 \\ i\Omega\, y_o\,(z)\, e^{i\Omega t} \end{Bmatrix} \qquad \text{at the tank sector wall } \varphi = 0 \quad (2.3)$$

$$\frac{1}{r}\frac{\partial \emptyset}{\partial \varphi} = \begin{Bmatrix} -i\Omega\, x_o\,(z)\, e^{i\Omega t} \sin 2\pi\alpha \\ i\Omega\, y_o\,(z)\, e^{i\Omega t} \cos 2\pi\alpha \end{Bmatrix} \qquad \text{at the tank sector wall } \varphi = 2\pi\alpha \qquad (2.4)$$

$$\frac{\partial^2 \emptyset}{\partial t^2} + g\frac{\partial \emptyset}{\partial z} = 0 \qquad \text{at the free fluid surface } z = 0 \qquad (2.5)$$

where the upper and lower line on the right-hand side represents excitation in x-direction and y-direction, respectively.

The Green function was used in this treatment, since in most cases the displacement curve of the bending structure is not analytically known, and since the representation of the solution as an integral is more advantageous for the numerical evaluation on high speed computers.

Separating the motion of the rigid body with solidified surface, the potential becomes

$$\emptyset = \psi + \begin{Bmatrix} i\Omega\, r\, x_o\,(z) \cos\varphi \\ i\Omega\, r\, y_o\,(z) \sin\varphi \end{Bmatrix} e^{i\Omega t}\,. \qquad (2.6)$$

We obtain for the boundary conditions of the disturbance potential the expressions

$$\frac{\partial \psi}{\partial r} = 0 \qquad \text{for } r = a, b; \qquad (2.7)$$

$$\frac{1}{r}\frac{\partial\psi}{\partial\varphi} = 0 \qquad \text{for } \varphi = 0,\ 2\pi\alpha \tag{2.8}$$

$$\frac{\partial\psi}{\partial z} = \begin{Bmatrix} -i\Omega\, x_0'(z)\ r \cos\varphi \\ -i\Omega\, y_0'(z)\ r \sin\varphi \end{Bmatrix} \qquad \text{for } z = -h; \tag{2.9}$$

$$g\frac{\partial\psi}{\partial z} - \Omega^2\psi = \begin{Bmatrix} i\Omega\ r \cos\varphi[\Omega^2\, x_0(0) - g\, x_0'(0)] \\ i\Omega\ r \sin\varphi[\Omega^2\, y_0(0) - g\, y_0'(0)] \end{Bmatrix} \qquad \text{for } z = 0. \tag{2.10}$$

Instead of the Laplace equation, the Poisson equation of the form

$$\Delta\psi = \begin{Bmatrix} -i\Omega\, x_0''(z)\ r \cos\varphi \\ -i\Omega\, y_0''(z)\ r \sin\varphi \end{Bmatrix} \tag{2.11}$$

has to be solved with the above boundary conditions.

The solution of the Poisson equation which satisfies the first (2.7) and second (2.8) boundary conditions is of the form

$$\psi(r,\ \varphi,\ z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}(z)\ \cos\left(\frac{m}{2\alpha}\varphi\right) C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right) \equiv A(z)\ \cos\bar{\varphi}\ C(\rho) \tag{2.12}$$

where

$$C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right) \equiv C(\rho) = J_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right) Y'_{\frac{m}{2\alpha}}\left(\xi_{mn}\right) - J'_{\frac{m}{2\alpha}}\left(\xi_{mn}\right) Y_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right)$$

The values ξ_{mn} are the positive roots of the equation

$$\Delta_{\frac{m}{2\alpha}} = J'_{\frac{m}{2\alpha}}(\xi)\ Y'_{\frac{m}{2\alpha}}(k\xi) - J'_{\frac{m}{2\alpha}}(k\xi)\ Y'_{\frac{m}{2\alpha}}(\xi) = 0.$$

The expression k = b/a is the diameter ratio of the inner to the outer wall.

Here, for the sake of simplicity, the double summations and the indices m and n have been omitted. The abbreviations are:

$$\bar{\varphi} = \frac{m}{2\alpha}\varphi \; ; \quad \rho = \xi_{mn}\frac{r}{a}.$$

For the determination of the unknown coefficients A_{mn}, one expands the right-hand side of the tank bottom and free surface conditions into Fourier and Bessel series. The cosine and sine can be expressed as

$$\cos\varphi = \sum_{m=0}^{\infty} a_m \cos\bar{\varphi} \text{ with } a_o = \frac{\sin\bar{\alpha}}{\bar{\alpha}} \; ; \; a_m = \frac{2\alpha(-1)^m \sin\bar{\alpha}}{(m^2\pi^2 - \bar{\alpha}^2)} \; ; (\bar{\alpha} = 2\pi\alpha)$$

$$\sin\varphi = \sum_{m=0}^{\infty} c_m \cos\bar{\varphi} \text{ with } c_o = \frac{1-\cos\bar{\alpha}}{2} \; ; \; c_m = \frac{2\alpha[(-1)^m \cos\bar{\alpha} - 1]}{(m^2\pi^2 - \bar{\alpha}^2)}. \tag{2.13}$$

The radius r can be expressed in the form

$$r = \sum_{n=0}^{\infty} b_{mn}\, C(\rho) \qquad m = 0, 1, 2\ldots \tag{2.14}$$

in which the coefficients are

$$b_{mn} = \frac{a\int_{k\xi_{mn}}^{\xi_{mn}} \rho^2 C(\rho)\, d\rho}{\xi_{mn}\int_{k\xi_{mn}}^{\xi_{mn}} \rho C^2(\rho)\, d\rho} = \frac{2a\, N_2\left(\xi_{mn}\right)}{\left[\frac{4}{\pi^2\xi_{mn}^2} - k^2C^2\left(k\xi_{mn}\right)\right] - \frac{m}{4\alpha^2\xi_{mn}^2}\left[\frac{4}{\pi^2\xi_{mn}^2} - C^2\left(k\xi_{nm}\right)\right]}. \tag{2.15}$$

(See Appendix)

The solution of the Poisson equation can be obtained from the differential equations of the A_{mn}

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{A''_{mn}(z) - \frac{\xi_{mn}^2}{a^2} A_{mn}(z)\right\} C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right)\cos\left(\frac{m}{2\alpha}\varphi\right)$$

$$= \begin{Bmatrix} -i\Omega\, x''_0(z)\, r\cos\varphi \\ -i\Omega\, y''_0(z)\, r\sin\varphi \end{Bmatrix}. \tag{2.16}$$

With the above mentioned series expansion, we obtain an infinite number of ordinary differential equations

$$A''_{mn}(z) - \frac{\xi_{mn}}{a^2} A_{mn}(z) = \begin{Bmatrix} -i\Omega\, b_{mn}\, a_m\, x''_0(z) \\ -i\Omega\, b_{mn}\, c_m\, y''_0(z) \end{Bmatrix}. \tag{2.17}$$

$$(m, n = 0, 1, 2 \ldots)$$

These differential equations are solved with the Green function. From the boundary conditions at the tank bottom and free fluid surface, one obtains with the series expansions (2.13) and (2.14)

$$A'_{mn}(-h) = \begin{Bmatrix} -i\Omega\, b_{mn}\, a_m\, x'_0(-h) \\ -i\Omega\, b_{mn}\, c_m\, y'_0(-h) \end{Bmatrix}.$$

$$g\, A'_{mn}(0) - \Omega^2 A_{mn}(0) = \begin{Bmatrix} [\Omega^2 x_0(0) - g\, x'_0(0)]\, i\Omega b_{mn}\, a_m \\ [\Omega^2 y_0(0) - g\, y'_0(0)]\, i\Omega b_{mn}\, c_m \end{Bmatrix}.$$

A translational motion $x_0(-h)e^{i\Omega t}$ or $y_0(-h)e^{i\Omega t}$ is superimposed.

Taking $x_0(-h) = 0$ or $y_0(-h) = 0$ the clamped-in condition is obtained, while for $x'_0 = 0$ or $y'_0 = 0$ the translational excitation case results.

The Green function finally is

$$A_{imn} = G_i(z,\zeta) = \alpha_i(\zeta)\, e^{\frac{\xi_{mn} z}{a}} + \beta_i(\zeta)\, e^{-\frac{\xi_{mn} z}{a}}. \tag{2.18}$$

$$i = 1 \text{ for } -h \leq z \leq \zeta$$
$$i = 2 \text{ for } \zeta \leq z \leq 0$$

It is at $z = \zeta$

$$G_1(\zeta,\zeta) = G_2(\zeta,\zeta).$$

The first derivative of the Green function has at $z = \zeta$ a discontinuity of unity

$$G_1'(\zeta,\zeta) - G_2'(\zeta,\zeta) = -1.$$

With this we conclude that

$$\alpha_1 e^{\frac{\xi_{mn}\zeta}{a}} - \alpha_2 e^{\frac{\xi_{mn}\zeta}{a}} + \beta_1 e^{-\frac{\xi_{mn}\zeta}{a}} - \beta_2 e^{-\frac{\xi_{mn}\zeta}{a}} = 0.$$

$$\alpha_1 \frac{\xi_{mn}}{a} e^{\frac{\xi_{mn}\zeta}{a}} - \alpha_2 \frac{\xi_{mn}}{a} e^{\frac{\xi_{mn}\zeta}{a}} - \beta_1 \frac{\xi_{mn}}{a} e^{-\frac{\xi_{mn}\zeta}{a}} + \beta_2 \frac{\xi_{mn}}{a} e^{-\frac{\xi_{mn}\zeta}{a}} = -1$$

and with the two homogeneous boundary conditions

$$\alpha_1 \frac{\xi_{mn}}{a} e^{-\frac{\xi_{mn}h}{a}} - \beta_1 \frac{\xi_{mn}}{a} e^{\frac{\xi_{mn}h}{a}} = 0.$$

$$\left(\frac{g\,\xi_{mn}}{a} - \Omega^2\right)\alpha_2 - \left(\frac{g\,\xi_{mn}}{a} + \Omega^2\right)\beta_2 = 0.$$

From these four linear equations of the unknown values α_1, α_2, β_1, β_2 one obtains

$$\alpha_1 = -\frac{\eta^2 e^{\frac{\xi_{mn}h}{a}} \left\{\sinh\left(\xi_{mn}\frac{\zeta}{a}\right) + \frac{\xi_{mn}\,g}{a\Omega^2}\cosh\left(\frac{\xi_{mn}}{a}\zeta\right)\right\}}{2\frac{\xi_{mn}}{a}\cosh\left(\xi_{mn}\frac{h}{a}\right)\cdot(1-\eta^2)}$$

$$\beta_1 = - \frac{\eta^2 e^{-\frac{\xi_{mn} h}{a}} \left\{ \frac{g\, \xi_{mn}}{a^2\, \Omega^2} \cosh\left(\xi_{mn} \frac{\zeta}{a}\right) + \sinh\left(\xi_{mn} \frac{\zeta}{a}\right) \right\}}{2\, \xi_{mn} \cosh\left(\xi_{mn} \frac{h}{a}\right) (1 - \eta^2)}$$

$$\alpha_2 = - \frac{\left(1 + \frac{\xi_{mn}\, g}{a\Omega^2}\right) \eta^2 \cosh\left[\frac{\xi_{mn}}{a} (\zeta + h)\right]}{2\, \frac{\xi_{mn}}{a} \cosh\left(\xi_{mn} \frac{h}{a}\right) (1 - \eta^2)}$$

$$\beta_2 = - \frac{\left(\frac{\xi_{mn}\, g}{a\Omega^2} - 1\right) \eta^2 \cosh\left[\frac{\xi_{mn}}{a} (\zeta + h)\right]}{2\, \frac{\xi_{mn}}{a} \cosh\left(\xi_{mn} \frac{h}{a}\right) (1 - \eta^2)}$$

where $\eta = \frac{\Omega}{\omega_{mn}}$ is the ratio of exciting-to Eigen-frequency, and

$$\omega_{mn}^2 = \frac{g\, \xi_{mn}}{a} \tanh\left(\xi_{mn} \frac{h}{a}\right).$$

The Green function is then

$$G_1(z,\zeta) = - \frac{\left\{\sinh\left(\frac{\xi_{mn}}{a}\, \zeta\right) + \frac{\xi_{mn}\, g}{a\Omega^2} \cosh\left(\xi_{mn} \frac{\zeta}{a}\right)\right\}}{\frac{\xi_{mn}}{a} \cosh\left(\xi_{mn} \frac{h}{a}\right) (1 - \eta^2)} \eta^2 \cosh\left[\xi_{mn}\left(\frac{z}{a} + \frac{h}{a}\right)\right]$$

for $-h \leq z \leq \zeta$. (2.19).

$$G_{\mathfrak{z}}(z,\zeta) = -\frac{\left\{\sinh\left(\xi_{mn}\frac{z}{a}\right) + \frac{\xi_{mn}\,g}{a\Omega^2}\cosh\left(\xi_{mn}\frac{z}{a}\right)\right\}}{\frac{\xi_{mn}}{a}\cosh\left(\xi_{mn}\frac{h}{a}\right)(1-\eta^2)}\,\eta^2\cosh\left[\xi_{mn}\left(\frac{\xi}{a}+\frac{h}{a}\right)\right]$$

for $\zeta \leq z \leq 0.$ (2.20)

The solution for the homogeneous boundary conditions in then:

$$A_{mn}(z) = i\Omega b_{mn}\begin{Bmatrix} a_m \\ \\ c_m \end{Bmatrix}\frac{\left[\sinh\left(\xi_{mn}\frac{\zeta}{a}\right) + \frac{\xi_{mn}\,g}{a\Omega^2}\cosh\left(\xi_{mn}\frac{\zeta}{a}\right)\right]}{\frac{\xi_{mn}}{a}\cosh\left(\frac{\xi_{mn}}{a}h\right)(1-\eta^2)}\,\eta^2.$$

$$\int_{-h}^{z}\begin{Bmatrix} x\mathfrak{d}(\zeta) \\ y\mathfrak{d}(\zeta)\end{Bmatrix}\cosh\left[\frac{\xi_{mn}}{a}(\zeta+h)\right]d\zeta +$$

$$+ i\Omega b_{mn}\begin{Bmatrix} a_m \\ c_m \end{Bmatrix}\frac{\cosh\left[\frac{\xi_{mn}}{a}\left(\frac{z}{a}+\frac{h}{a}\right)\right]}{\frac{\xi_{mn}}{a}\cosh\left(\xi_{mn}\frac{h}{a}\right)(1-\eta^2)}\,\eta^2\int_{z}^{0}\begin{Bmatrix} x\mathfrak{d}(\zeta) \\ y\mathfrak{d}(\zeta)\end{Bmatrix}\left[\sinh\left(\xi_{mn}\frac{\zeta}{a}\right) +\right.$$

$$\left. + \frac{g\xi_{mn}}{a\Omega^2}\cosh\left(\xi_{mn}\frac{\zeta}{a}\right)\right]d\zeta. \qquad (2.21)$$

The solution for the inhomogeneous boundary condition

$$A_{mn}(-h) = 1$$

$$g\,A'_{mn}(0) - \Omega^2 A_{mn}(0) = 0$$

is

$$i\Omega b_{mn}\ \eta^2 \begin{Bmatrix} a_m \\ c_m \end{Bmatrix} \frac{\left[\frac{\xi_{mn}\ g}{a\Omega^2} \cosh\left(\xi_{mn}\ \frac{z}{a}\right) + \sinh\left(\xi_{mn}\ \frac{z}{a}\right)\right]}{\frac{\xi_{mn}}{a} \cosh\left(\xi_{mn}\ \frac{h}{a}\right)\ (1 - \eta^2)} \tag{2.22}$$

and the one for the boundary conditions

$$A'_{mn}\ (-h) = 0$$

$$gA'_{mn}\ (0) - \Omega^2\ A_{mn}(0) = 1$$

is

$$i\Omega b_{mn}\ \eta^2 \begin{Bmatrix} a_m[x_0(0) - \frac{g}{\Omega^2}\ x_0'(0)] \\ c_m[y_0(0) - \frac{g}{\Omega^2}\ y_0'(0)] \end{Bmatrix} \cdot \frac{\cosh\left(\xi_{mn}\left(\frac{z}{a} + \frac{h}{a}\right)\right)}{\cosh\left(\xi_{mn}\ \frac{h}{a}\right)(1 - \eta^2)}. \tag{2.23}$$

The solution of the differential equation (2.17) is then:

$$A_{mn}(z) = \frac{i\Omega\ ab_{mn} \begin{Bmatrix} a_m \\ c_m \end{Bmatrix} \eta^2}{\xi_{mn}\ \cosh\left(\xi_{mn}\ \frac{h}{a}\right)\ (1 - \eta^2)} \cdot \left[\sinh\left(\xi_{mn}\ \frac{z}{a}\right) + \right.$$

$$\left. + \frac{\xi_{mn}\ g}{a\Omega^2} \cosh\left(\xi_{mn}\ \frac{z}{a}\right)\right] \cdot \left[\begin{Bmatrix} x_0'(-h) \\ y_0'(-h) \end{Bmatrix} + \right.$$

$$\left. + \int_{-h}^{z} \begin{Bmatrix} x_0''(\zeta) \\ y_0''(\zeta) \end{Bmatrix} \cosh\left[\xi_{mn}\left(\frac{\zeta}{a} + \frac{h}{a}\right)\right] d\zeta\right] + \cosh\left[\xi_{mn}\left(\frac{z}{a} + \frac{h}{a}\right)\right].$$

$$\cdot \left[\int_z^0 \begin{Bmatrix} x_0''(\zeta) \\ y_0''(\zeta) \end{Bmatrix} \left[\sinh\left(\xi_{mn}\frac{\zeta}{a}\right) + \frac{\xi_{mn}\, g}{a\Omega^2} \cosh\left(\xi_{mn}\frac{\zeta}{a}\right) \right] d\zeta + \right.$$

$$\left. + \frac{\xi_{mn}}{a} \begin{Bmatrix} [x_0(0) - \frac{g}{\Omega^2} x_0'(0)] \\ [y_0(0) - \frac{g}{\Omega^2} y_0'(0)] \end{Bmatrix} \right] \Bigg\} . \tag{2.24}$$

The velocity potential is then

$$\phi(r, \varphi, z, t) = e^{i\Omega t} \left[\begin{Bmatrix} i\Omega x_0(z)\; r \cos\varphi \\ i\Omega y_0(z)\; r \sin\varphi \end{Bmatrix} + \right.$$

$$\left. + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}(z)\, C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right) \cos\left(\frac{m}{2\alpha}\varphi\right) \right] . \tag{2.25}$$

The term in front of the double summation satisfies the boundary conditions at the tank walls, while the terms of the double series vanish at the tank walls. The double summation together with the term in front of it satisfies the free surface condition, if one considers the results of (2.13), (2.14), and the Eigen values $\omega_{mn}^2 = \frac{g\,\xi_{mn}}{a} \tanh\left(\xi_{mn}\frac{h}{a}\right)$. With this velocity potential, the free surface displacement, the pressure and velocity distribution, as well as the forces and moments of the liquid can be obtained by differentiations and integration with respect to the time and spacial coordinates.

The surface displacement of the liquid measured from the undisturbed position

$$\bar{z} = \frac{\Omega^2}{g} e^{i\Omega t} \left[\begin{Bmatrix} x_0(0)\; r\cos\varphi \\ y_0(0)\; r\sin\varphi \end{Bmatrix} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}(0)\, C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right) \cos\left(\frac{m}{2\alpha}\varphi\right) \right] . \tag{2.26}$$

The pressure in a depth (-z) is

$$p = -\bar{\rho}\frac{\partial\phi}{\partial t} - g\bar{\rho}z = \bar{\rho}\Omega^2 e^{i\Omega t}\left[\begin{Bmatrix} x_0(z)\ r\cos\varphi \\ y_0(z)\ r\sin\varphi \end{Bmatrix} + \right.$$

$$\left. + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}(z)\, C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right)\cos\left(\frac{m}{2\alpha}\varphi\right)\right] - \bar{\rho}gz. \tag{2.27}$$

At the outer tank wall $r = a$ the value of $C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right)$ is $2/\pi\ \xi_{mn}$, while

at the inner tank wall $r = b$ it is $C_{\frac{m}{2\alpha}}\left(k\ \xi_{mn}\right)$.

At the sector walls $\varphi = 0$ and $\varphi = 2\pi\alpha$, the cosine function has the value 1 and $(-1)^m$, respectively. The pressure distribution at the tank bottom results from (2.27) by setting $z = -h$.

From the pressure distribution, the liquid forces and moments are determined by integrating the appropriate components. In x-direction, the force is:

$$F_x = \int_0^{2\pi\alpha}\int_{-h}^{0} (a\ p_a - bp_b)\cos\varphi\ d\varphi\ dz - \int_b^a\int_{-h}^{0} p_{\varphi = 2\pi\alpha}\sin 2\pi\alpha\ dr\ dz.$$

The first integral represents the contribution of the pressure distribution from the circular walls, while the remaining integral is due to the pressure distribution at the sector walls. With the liquid mass $m = \bar{\rho}\pi\ a^2\ h(1 - k^2)$, the liquid force in x-direction is:

$$F_x = m\Omega^2 e^{i\Omega t}\left[\begin{Bmatrix} \frac{1}{h}\int_{-h}^{0} x_0(z)\ dz \\ 0 \end{Bmatrix} + \right.$$

$$+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+1} \sin 2\pi\alpha}{i\Omega\pi\alpha\ a\ h\ (1 - k^2)} \int_{-h}^{o} A_{mn}(z)\ dz.$$

$$\left. \left(\frac{4\alpha^2}{(m^2 - 4\alpha^2)} \left[\frac{2}{\pi\ \xi_{mn}} - k\ C_{\frac{m}{2\alpha}}\ (k\ \xi_{mn}) \right] + N_o(\xi_{mn}) \right) \right] \tag{2.28}$$

where

$$N_o(\xi_{mn}) = \frac{1}{\xi_{mn}} \int_{k\ \xi_{mn}}^{\xi_{mn}} C\ (\rho)\ d\rho. \qquad \text{(See Appendix.)}$$

The force component in y-direction is with

$$F_y = \int_{0}^{2\pi\alpha} \int_{-h}^{o} [a\ p_a - b\ p_b] \sin\varphi\ dz\ d\varphi - \int_{b}^{a} \int_{-h}^{o} p_{\varphi=0}\ dr\ dz$$

$$+ \int_{b}^{a} \int_{-h}^{o} p_{\varphi\ =\ 2\pi\alpha} \cos 2\pi\alpha\ dr\ dz$$

given by:

$$F_y = m\Omega^2\ e^{i\Omega t} \left[\left\{ \begin{matrix} 0 \\ \\ \frac{1}{h} \int_{-h}^{o} y_o(z)\ dz \end{matrix} \right\} \right. +$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{i\,[1-(-1)^m \cos 2\pi\alpha]}{2\pi\alpha\, a\, h(1-k^2)} \int_{-h}^{o} A_{mn}(z)\, dz.$$

$$\cdot \left(\frac{4\alpha^2}{(m^2-4\alpha^2)} \left[\frac{2}{\pi\, \xi_{mn}} - k\, C_{\frac{m}{2\alpha}} (k\, \xi_{mn})\right] + N_o(\xi_{mn})\right)\Bigg] . \tag{2.29}$$

The term $m\Omega^2 \frac{e^{i\Omega t}}{h} \int_{-h}^{o} x_o(z)\, dz$ (in front of the double summation) in F_x represents the inertial force, i.e., the liquid force with restraint surface. The liquid moments with respect to the point $(0,0,-\frac{h}{2})$ are given by

$$M_y = \int_{o}^{2\pi\alpha}\int_{-h}^{o} (a\, p_a - b\, p_b) \cdot (\frac{h}{2}+z) \cos\varphi\, d\varphi\, dz +$$

$$+\int_{o}^{2\pi\alpha}\int_{b}^{a} p_c\, r^2 \cos\varphi\, d\varphi\, dr - \int_{b}^{a}\int_{-h}^{o} p_{\varphi=2\pi\alpha} \sin 2\pi\alpha\, (\frac{h}{2}+z)\, dr\, dz$$

and

$$M_x = -\int_{o}^{2\pi\alpha}\int_{-h}^{o} (a\, p_a - b\, p_b)(\frac{h}{2}+z) \sin\varphi\, d\varphi\, dz - \int_{o}^{2\pi\alpha}\int_{b}^{a} p_c\, r^2 \sin\varphi\, d\varphi\, dr +$$

$$+\int_{b}^{a}\int_{-h}^{o} (\frac{h}{2}+z)\, p_{\varphi=0}\, dr\, dz - \int_{b}^{a}\int_{-h}^{o} (\frac{h}{2}+z)\, p_{\varphi=2\pi\alpha} \cos 2\pi\alpha\, dr\, dz.$$

M_y is the moment about the parallel axis to the y-axis through the point $(0, 0, -\frac{h}{2})$ and M_x is the moment about a parallel axis to the x-axis through the same point. The first integral represents the contribution of the pressure distribution from the circular walls. The second integral is the contribution of the pressure at the tank bottom, while the remaining integrals can be easily identified as the contribution to the moment due to the pressure distribution at the tank sector walls. The moments are given by

$$
M_y = m\Omega^2 e^{i\Omega t} \left[\left\{ \begin{matrix} \left(\frac{1}{h} \int_{-h}^{0} (\frac{h}{2} + z) x_0(z) dz + \frac{a^2 x_0(-h)(1 + k^2)}{4h} \left(1 + \frac{\sin 2\pi\alpha \cos 2\pi\alpha}{2\pi\alpha} \right) \right) \\ \\ \left(0 \qquad + \frac{a^2 y_0(-h)(1 + k^2)}{4h} \quad \frac{\sin^2 2\pi\alpha}{2\pi\alpha} \right) \end{matrix} \right\} \right.
$$

$$
+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} \sin 2\pi\alpha}{i\Omega\pi\alpha\, a\, h(1 - k^2)} \left\{ \frac{4\alpha^2 a^2}{(m^2 - 4\alpha^2)} N_2^{(\frac{m}{2\alpha})} (\xi_{mn})\, A_{mn}^{\binom{(x)}{(y)}} (-h) + \right.
$$

$$
+ \left[\frac{4\alpha^2}{(m^2 - 4\alpha^2)} \left[\frac{2}{\pi \xi_{mn}} - k\, C_{\frac{m}{2\alpha}} (k\, \xi_{mn}) \right] + N_0^{(\frac{m}{2\alpha})} (k\, \xi_{mn}) \right] \cdot
$$

$$
\left. \left. \int_{-h}^{0} (\frac{h}{2} + z)\, A_{mn}^{\binom{(x)}{(y)}} (z)\, dz \right\} \right] + mg\, \frac{a}{3}\, \frac{\sin 2\pi\alpha}{\pi\alpha}\, \frac{(1 + k + k^2)}{(1 + k)}
$$

(2.30)

and

$$M_x = m\Omega^2 e^{i\Omega t} \left[\left\{ \frac{a^2(1+k^2)x_C(-h)}{4h} \frac{\sin^2 2\pi\alpha}{2\pi\alpha} \right. \right.$$

$$\left. \frac{1}{h} \int_{-h}^{0} \left(\frac{h}{2} + z\right) y_O(z)dz + \frac{a^2(1+k^2)y_O(-h)}{4h} \left(1 - \frac{\sin 2\pi\alpha \cos 2\pi\alpha}{2\pi\alpha}\right) \right\} +$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{i[1 - (-1)^m \cos 2\pi\alpha]}{\Omega\, a\, \pi\alpha\, h(1-k^2)} \cdot \left\{ \frac{4\alpha^2 a^2}{(m^2 - 4\alpha^2)} A_{mn}^{(x)(y)} (-h)\, N_2^{(\frac{m}{2\alpha})} (\xi_{mn}) + \right.$$

$$+ \left[\frac{4\alpha^2}{m^2 - 4\alpha^2} \left(\frac{2}{\pi\, \xi_{mn}} - k\, C_{\frac{m}{2\alpha}} (k\, \xi_{mn}) + N_o^{(\frac{m}{2\alpha})} (\xi_{mn}) \right] \right.$$

$$\left. \left. \int_{-h}^{0} \left(\frac{h}{2} + z\right) A_{mn}^{(x)(y)}(z)\, dz \right\} \right] + mg\, \frac{a}{3} \frac{[1 - \cos 2\pi\alpha]}{\pi\alpha} \frac{(1 + k + k^2)}{(1 + k)} \tag{2.31}$$

where $N^2(\xi_{mn}) = \frac{1}{\xi_{mn}^{\ 3}} \int_{k\,\xi_{mn}}^{\xi_{mn}} \rho^2\, C(\rho)\, d\rho$. The last term in these results represents the moment of the undisturbed liquid about the point $(0, 0, -\frac{h}{2})$. The velocity distribution is:

$$u_r = e^{i\Omega t}\left[\begin{Bmatrix} i\Omega\, x_o(z)\cos\varphi \\ \\ i\Omega\, y_o(z)\sin\varphi \end{Bmatrix} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn}^{\substack{(x)\\(y)}}(z)\cos\left(\frac{m}{2\alpha}\varphi\right)\frac{\xi_{mn}}{a}\, C'_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right)\right]$$

$$u_\varphi = -\, e^{i\Omega t}\left[\begin{Bmatrix} i\Omega\, x_o(z)\sin\varphi \\ \\ -i\Omega\, y_o(z)\cos\varphi \end{Bmatrix} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A'^{\substack{(x)\\(y)}}_{mn}(z)\sin\left(\frac{m}{2\alpha}\varphi\right)\frac{m}{2\alpha r}\, C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right)\right]$$

$$w = e^{i\Omega t}\left[\begin{Bmatrix} i\Omega\, r\, x_o'(z)\cos\varphi \\ \\ i\Omega\, r\, y_o'(z)\sin\varphi \end{Bmatrix} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A'^{\substack{(x)\\(y)}}_{mn}(z)\cos\left(\frac{m}{2\alpha}\varphi\right) C_{\frac{m}{2\alpha}}\left(\xi_{mn}\frac{r}{a}\right)\right]$$

The velocity distribution in the tank is obtained by omitting the first term in u_r and u_φ. These terms represent nothing but the tank motion. Making $x_o(z) = x_o$ = constant or $y_o(z) = y_o$ = constant, the results for translational oscillations will be obtained [1].

III. SPECIAL CASES

Tanks with circular cross section are at present the most frequent containers. Tendencies in space technology, however, point toward the intersection of tank by radial walls. Subdivision of a cylindrical tank into four quarter compartments is a possibility to reduce the dynamic influence of the propellant sloshing upon the stability of the space vehicle. Concentric containers could be beneficial, if one could choose their diameter ratio in such a way that the phasing and sloshing mass of the propellant inside and outside is such that their combined effect cancels. The theory of liquid motion due to bending tank wall was already presented for annular cylindrical tanks [2]. In the following we shall restrict ourselves to the sector and circular tank. The main results are given in Table I.

A. Sector Tank

If the diameter ratio $k = b/a$ tends to zero, then the results of (2.2) represent the motion of a liquid with free fluid surface in a tank with circular sector cross section. The zeros of the determinant $\frac{\Delta m}{2\alpha}(\xi) = 0$ are now obtained from the equation $J'_m(\epsilon) = 0$ and are noted by ϵ_{mn}. Furthermore, the function $C(\rho)$ is substituted by $J(\rho) \equiv J_{\frac{m}{2\alpha}}(\epsilon_{mn}\frac{r}{a})$. The integration constants b_{mn} transform into

$$\bar{b}_{mn} = \frac{a \int_0^{\epsilon_{mn}} \rho^2 J(\rho)\, d\rho}{\epsilon_{mn} \int_0^{\epsilon_{mn}} \rho J^2(\rho)\, d\rho} \qquad m, n = 0, 1, 2, \ldots\ldots$$

which is

$$\bar{b}_{mn} = 2a \frac{\frac{\Gamma(m/4\alpha + 3/2)}{\Gamma(m/4\alpha - 1/2)} \sum_{\mu=0}^{\infty} \frac{(m/2\alpha + 2\mu + 1)\,\Gamma(m/4\alpha + \mu - 1/2)}{\Gamma(m/4\alpha + \mu + 5/2)} J_{m/2\alpha + 2\mu + 1}(\epsilon_{mn})}{\epsilon_{mn}(1 - m^2/4\alpha^2 \epsilon^2_{mn})\, J^2_{\frac{m}{2\alpha}}(\epsilon_{mn})} \tag{3.1}$$

In the force components, one has to omit the singular solution at $r = 0$. Therefore, the value $2/\pi\, \xi_{mn} - k\, C_{\frac{m}{2\alpha}}(k\, \xi_{mn})$ has to be substituted by $J_{\frac{m}{2\alpha}}(\epsilon_{mn})$. The expression $N_o(\xi_{mn})$ transforms into $L_o(\epsilon_{mn})$, since $\int C(\rho)\, d\rho$ is $\int J(\rho)\, d\rho$. Similar results are valid for $N_2(\xi_{mn})$, which transforms into $L_2(\xi_{mn})$. The velocity potential, the free fluid surface displacement and the force and moment components are represented in Table I.

B. Circular Cylindrical Tank

For a container with circular cross section it is $\alpha = 1$. This represents a container with a side wall in the $\varphi = 0$ plane from $r = 0$ to $r = a$. The values a_m, $\bar{b}_{mn}$ are then:

$$a_o = a_m = 0, \quad a_2 = \lim_{\alpha \to 1} \left\{ - \frac{\alpha \sin 2\pi\alpha}{\pi(1 - \alpha^2)} \right\} = 1.$$

If one chooses an excitation in x-direction, the sidewall does not disturb the flow field. The expression $\bar{b}_{2n}$ is obtained from (3.1) with the recurrence formula of the Bessel functions $x\,J_v'(x) - v\,J_v(x) = -x\,J_{v-1}(x)$, which is for $x = \epsilon_n$ as the zeros of the equation $J_1(\epsilon_n) = 0$ and with the consideration of the singularity of the Gamma Function at the argument zero,

$$\bar{b}_{2n} = \frac{2a}{(\epsilon_n^2 - 1)\, J_1(\epsilon_n)} \,.$$

The velocity potential is therefore due to bending excitation in x-direction:

$$\phi(r, \varphi, z, t) = i\Omega e^{i\Omega t}\, a \cos\varphi \left\{ \frac{r}{a}\, x_o(z) + \right.$$

$$+ 2 \sum_{n=1}^{\infty} \frac{J_1\left(\epsilon_n \frac{r}{a}\right) \eta^2}{(\epsilon_n^2 - 1)\, J_1(\epsilon_n)\, \frac{\epsilon_n}{a} \cosh\left(\epsilon_n \frac{h}{a}\right) (1 - \eta^2)}$$

$$\cdot \left[\left(\sinh\left(\epsilon_n \frac{z}{a}\right) + \frac{\epsilon_n g}{a\Omega^2} \cosh\left(\epsilon_n \frac{z}{a}\right) \right) \cdot \left(x_o'(-h) + \right. \right.$$

$$\left. + \int_{-h}^{z} x_o''(\zeta) \cosh\left[\frac{\epsilon_n}{a} (\zeta + h) \right] d\zeta \right) + \cosh\left[\frac{\epsilon_n}{a} (z + h) \right] .$$

$$\cdot\left(\int_{z}^{o} x\gamma(\zeta)\left[\sinh\left(\epsilon_n \frac{\zeta}{a}\right) + \frac{\epsilon_n g}{a\,\Omega^2}\cosh\left(\frac{\epsilon_n \zeta}{a}\right)\right] d\zeta \; + \right.$$

$$\left. + \frac{\epsilon_n}{a}\left[x_0(0) - \frac{g}{\Omega^2}\dot{x_0}(0)\right]\right)\Bigg]\Bigg\} \; . \qquad (3.2)$$

The free fluid surface displacement and the force and moment of the liquid are represented in Table 2.

It can be seen from the free fluid surface displacement that with increasing exciting amplitude $x_0(0)$ at the free fluid surface location the surface displacement increases. With increasing acceleration the displacement of the free fluid surface decreases.

In the liquid force the limit value is

$$\lim_{\substack{\alpha \to 1 \\ m \to 2}} \left\{(-1)^{m+1} \frac{\sin 2\pi\alpha}{\pi\alpha}\left[\frac{4\alpha^2}{m^2 - 4\alpha^2} J_{\frac{m}{2\alpha}}(\epsilon_{mn}) + L_o(\epsilon_{mn})\right]\right\} = J_1(\epsilon_n)$$

and in the moment the value $L_2(\epsilon_n) = J_1(\epsilon_n)/\epsilon_n^2$ and

$$\lim_{\substack{\alpha \to 1 \\ m \to 2}} \left[(-1)^{m+1} \frac{\sin 2\pi\alpha}{\pi\alpha} \frac{8\alpha^2}{(m^2 - 4\alpha^2)}\right] = 2.$$

In the numerical evaluation, a bending displacement is considered, which has the form

$$x_0(z) = -\frac{4a}{250}\left[\left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right) - 6\right].$$

Here a tank of height $H = 5a$ is chosen (Figure 2). The maximum amplitude of this displacement is $x_{o/a} = 1/10$.

Figure 3 exhibits the fluid force versus fluid height ratio h/a for various exciting frequency ratios $\eta_1 = \Omega/\omega_1$. The fluid height has, of course, considerable influence, since the exciting amplitude is changing along the z-axis. The moment of the liquid (Figure 3) exhibits similar behavior.

The total force and moment are generally less for a given maximum bending amplitude than for a translational motion of the same magnitude. The maximum dynamic effects occur when the free fluid surface is located around the point of maximum bending displacement. With increasing fluid height the contribution of the sloshing mass decreases, while the inertial force increases. With increasing exciting frequency this inertial force becomes larger. Close to liquid resonance therefore the inertial effect is not strong enough to overcome the decrease of the effect of the sloshing fluids due to the diminished local exciting amplitude with increasing fluid height. Pressure and velocity distribution can easily be obtained [3].

APPENDIX

A. Roots of Certain Bessel Functions. For the previous results the roots of

$$\Delta_{\frac{m}{2\alpha}}(\xi) = \begin{vmatrix} J'_{\frac{m}{2\alpha}}(\xi) & Y'_{\frac{m}{2\alpha}}(\xi) \\ J'_{\frac{m}{2\alpha}}(k\xi) & Y'_{\frac{m}{2\alpha}}(k\xi) \end{vmatrix} = 0$$

have to be determined for $m = 0, 1, 2, \ldots\ldots.$ and arbitrary $0 \leq k < 1$. For most of these roots J. McMahon represented asymptotic expansions [4]. The smallest root, however, was not known until H. Buchholz pointed out its existence [5]. D. Kirkham [6] gave the roots of the above equation in a graphical way for $m/2\alpha = 0, 1, 2, 3, 4$.

B. Representation of a Function in Bessel-Fourier-Series

The determinant $C_{\frac{m}{2\alpha}}$ is

$$C_{\frac{m}{2\alpha}}(\lambda_{mn} r) = \begin{vmatrix} J_{\frac{m}{2\alpha}}(\lambda_{mn} r) & Y_{\frac{m}{2\alpha}}(\lambda_{mn} r) \\ J'_{\frac{m}{2\alpha}}(\lambda_{mn} a) & Y'_{\frac{m}{2\alpha}}(\lambda_{mn} a) \end{vmatrix}$$

Its derivative is

$$C'_{\frac{m}{2\alpha}}(\lambda_{mn} r) = \begin{vmatrix} J'_{\frac{m}{2\alpha}}(\lambda_{mn} r) & Y'_{\frac{m}{2\alpha}}(\lambda_{mn} r) \\ J'_{\frac{m}{2\alpha}}(\lambda_{mn} a) & Y'_{\frac{m}{2\alpha}}(\lambda_{mn} a) \end{vmatrix}$$

which vanishes for r = a and r = b, that is,

$$C'_{\frac{m}{2\alpha}}(\lambda_{mn} a) = C'_{\frac{m}{2\alpha}}(\lambda_{mn} b) = 0$$

for r = a, the derivative of $C_{\frac{m}{2\alpha}}$

vanishes identically, while for r = b the roots $(\xi_{mn}) = (\lambda_{mn} a)$ make it vanish.

A function f(r), which is piecewise regular in the interval

$b \leq r \leq a$,

satisfies the Dirichlet condition and can be expanded into a Bessel-Fourier series of the form

$$f(r) = \sum_{n=0}^{\infty} b_{mn}^{(f)} C_{\frac{m}{2\alpha}}(\lambda_{mn} r). \qquad (m = 0, 1, 2, \ldots.)$$

The unknown coefficients of the expansion will be determined by multiplying both sides of the equation with r $C_{\frac{m}{2\alpha}}(\lambda_{mp} r)$ and integrating from

$r = b$ to $r = a$. Here λ_{mp} and λ_{mn} are different roots of the determinant $\Delta_{\frac{m}{2\alpha}} = 0$. It is

$$\sum_{n=0}^{\infty} b_{mn} \int_a^b r\, C_{\frac{m}{2\alpha}}(\lambda_{mn} r)\, C_{\frac{m}{2\alpha}}(\lambda_{mp} r)\, dr = \int_b^a r\, f(r)\, C_{\frac{m}{2\alpha}}(\lambda_{mp} r)\, dr.$$

With the integral of Lommel, we obtain

$$(\lambda_{mn}^2 - \lambda_{mp}^2) \int r\, C_{\frac{m}{2\alpha}}(\lambda_{mn} r)\, C_{\frac{m}{2\alpha}}(\lambda_{mp} r)\, dr = r \left\{ C_{\frac{m}{2\alpha}}(\lambda_{mn} r) \frac{dC_{\frac{m}{2\alpha}}(\lambda_{mp} r)}{dr} - \right.$$

$$\left. - C_{\frac{m}{2\alpha}}(\lambda_{mp} r) \frac{dC_{\frac{m}{2\alpha}}(\lambda_{mn} r)}{dr} \right\} (n \neq p)$$

and the integral on the left-hand side is

$$\int r\, C_{\frac{m}{2\alpha}}(\lambda_{mn} r)\, C_{\frac{m}{2\alpha}}(\lambda_{mp} r)\, dr = \frac{r}{(\lambda_{mn}^2 - \lambda_{mp}^2)} \left\{ \lambda_{mp}\, C_{\frac{m}{2\alpha}}(\lambda_{mn} r) \right.$$

$$\left. C'_{\frac{m}{2\alpha}}(\lambda_{mp} r) - \lambda_{mn}\, C_{\frac{m}{2\alpha}}(\lambda_{mp} r)\, C'_{\frac{m}{2\alpha}}(\lambda_{mn} r) \right\}. \quad (p \neq n) \tag{F}$$

The integral is zero, if the conditions are satisfied:

$$C'_{\frac{m}{2\alpha}}(\lambda_{mn} a) = C'_{\frac{m}{2\alpha}}(\lambda_{mp} a) = C'_{\frac{m}{2\alpha}}(\lambda_{mn} b) = C'_{\frac{m}{2\alpha}}(\lambda_{mp} b) = 0. \tag{G}$$

Those terms for which $\lambda_{mn} \neq \lambda_{mp}$ vanish, and one obtains for the coefficients

$$b_{mn}^{(f)} = \frac{\int_b^a r\, f(r)\, C_{\frac{m}{2\alpha}}(\lambda_{mn} r)\, dr}{\int_b^a r\, C^2_{\frac{m}{2\alpha}}(\lambda_{mn} r)\, dr}. \tag{H}$$

For p = n the equation (F) will be an indeterminate form, which will be treated with Taylor expansion or the rule of L'Hospital, and is with the Bessel differential equation for $C_{\frac{m}{2\alpha}}$

$$\int r\, C_{\frac{m}{2\alpha}}(\lambda_{mn} r)\, dr = \frac{r^2}{2}\left\{ C^2_{\frac{m}{2\alpha}}(\lambda_{mn} r)\left[1 - \frac{m^2}{4\alpha^2 \lambda_{mn}^2 r^2}\right] + C'^2_{\frac{m}{2\alpha}}(\lambda_{mn} r)\right\}. \tag{I}$$

We thus obtain in the interval $b \leq r \leq a$

$$\int_b^a r\, C^2_{\frac{m}{2\alpha}}(\lambda_{mn} r)\, dr = \left\{ \begin{array}{l} \frac{a^2}{2}\left[C^2_{\frac{m}{2\alpha}}(\lambda_{mn} a)\left(1 - \frac{m^2}{4\alpha^2 \lambda_{mn}^2 a^2}\right) + C'^2_{\frac{m}{2\alpha}}(\lambda_{mn} a)\right] \\ \\ - \frac{b^2}{2}\left[C^2_{\frac{m}{2\alpha}}(\lambda_{mn} b)\left(1 - \frac{m^2}{4\alpha^2 \lambda_{mn}^2 b^2}\right) + C'_{\frac{m}{2\alpha}}(\lambda_{mn} b)\right] \end{array} \right\} \tag{J}$$

which is due to the boundary conditions

$$\int_b^a r\, C^2_{\frac{m}{2\alpha}}\left(\xi_{mn} \frac{r}{a}\right) dr = \frac{a^2}{2\xi_{mn}^2}\left[\frac{4}{\pi^2 \xi_{mn}^2}\left(\xi_{mn}^2 - \frac{m^2}{4\alpha^2}\right) - C^2_{\frac{m}{2\alpha}}(k\, \xi_{mn})\left(k^2 \xi_{mn}^2 - \frac{m^2}{4\alpha^2}\right)\right]. \tag{K}$$

Here, $C_{\frac{m}{2\alpha}}(\xi_{mn}) = \frac{2}{\pi\,\xi_{mn}}$ is the Wronskian determinant. The coefficient $b_{mn}^{(f)}$ of the Bessel-Fourier expansion can be determined from

$$b_{mn}^{(f)} = \frac{2\,\xi_{mn}^2 \int_b^a r\, f(r)\, C_{\frac{m}{2\alpha}}(\xi_{mn}\frac{r}{a})\, dr}{a^2\left[\frac{4}{\pi^2\,\xi_{mn}^2}(\xi_{mn}^2 - \frac{m^2}{4\alpha^2}) - C^2_{\frac{m}{2\alpha}}(k\,\xi_{mn})\,(k^2\,\xi_{mn}^2 - \frac{m^2}{4\alpha^2})\right]}. \quad \text{(L)}$$

The problem that remains is the solution of the $\int_b^a r\, f(r)\, C_{\frac{m}{2\alpha}}(\xi_{mn}\frac{r}{a})\, dr$.

Most of the integrals in the previous treatment are of the form

$$\int z^{\kappa}\, C_{\nu}(z)\, dz.$$

These can be obtained with the help of the Lommel functions $S_{\kappa\nu}(z)$ or by integration of the series expansion of the integrals.

$$\int z^{\kappa}\, C_{\frac{m}{2\alpha}}(z)\, dz = Y'_{\frac{m}{2\alpha}}(\xi_{mn}) \int z^{\kappa}\, J_{\frac{m}{2\alpha}}(z)\, dz - J'_{\frac{m}{2\alpha}}(\xi_{mn}) \int z^{\kappa}\, Y_{\frac{m}{2\alpha}}(z)\, dz. \quad \text{(M)}$$

Integrating the first integral term by term and collecting terms of $J_{\nu+2\mu+1}$, one obtains

$$\int z^{\kappa}\, J_{\nu}(z)\, dz = \frac{z^{\kappa}\,\Gamma(\frac{\kappa+\nu+1}{2})}{\Gamma(\frac{\nu-\kappa+1}{2})} \sum_{\mu=0}^{\infty} \frac{(\nu+2\mu+1)\,\Gamma(\frac{\nu+\kappa+1}{2}+\mu)}{\Gamma(\frac{\nu+\kappa+3}{2}+\mu)}\, J_{\nu+2\mu+1} \quad \text{(N)}$$

where Re $(\kappa+\mu+1)$ must be > 0 if one integrates from $z = 0$ on.

The second integral is obtained by termwise integration of the series expansion of the Bessel function of second kind

It is for ($\frac{m}{2\alpha}$ integer)

$$\int z^{\kappa} Y_{\frac{m}{2\alpha}}(z)\, dz = -\frac{z^{\kappa+1}}{\pi} \sum_{\mu=0}^{\frac{m}{2\alpha}-1} \frac{(\frac{m}{2\alpha}-\mu-1)!\,(\frac{z}{2})^{2\mu-\frac{m}{2\alpha}}}{\mu!\,(\kappa+2\mu-\frac{m}{2\alpha}+1)} +$$

$$\frac{2z^{\kappa+1}}{\pi} \sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}\,(\frac{z}{2})^{\frac{m}{2\alpha}+2\mu}}{\mu!\,(\frac{m}{2\alpha}+\mu)!\,(\frac{m}{2\alpha}+\kappa+2\mu+1)^2} \cdot \Big\{ \ln\frac{z}{2} - \frac{1}{2}\psi(\mu+1) -$$

$$-\frac{1}{2}\psi(\mu+\frac{m}{2\alpha}+1)\Big\} - \frac{2z^{\kappa+1}}{\pi} \sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}\,(\frac{z}{2})^{\frac{m}{2\alpha}+2\mu}}{\mu!\,(\frac{m}{2\alpha}+\mu)!\,(\frac{m}{2\alpha}+\kappa+2\mu+1)^2} \tag{O}$$

where $\psi(z)$ represents the logarithmic derivative of the Gamma function

$$\psi(z) = \frac{d\,(\ln\Gamma(z))}{dz} = -\gamma + (z-1)\sum_{\lambda=0}^{\infty} \frac{1}{(\lambda+1)(z-\lambda)} \tag{P}$$

and γ is the Euler constant. With these results, we obtain the integrals as mentioned in the text.

$$\int_b^a r^2\, C_{\frac{m}{2\alpha}}(\xi_{mn}\frac{r}{a})\, dr = a^3\, N_2^{(\frac{m}{2\alpha})}(\xi_{mn}).$$

$$\int_b^a C_{\frac{m}{2\alpha}}(\xi_{mn}\frac{r}{a})\, dr = a\, N_0^{(\frac{m}{2\alpha})}(\xi_{mn}). \tag{Q}$$

It may be mentioned here that some of the integrals in which μ is $1 - \nu$ or $\nu + 1$ can be obtained from the recursion formulas

$$\int z^{1-\nu} C_\nu (z)\, dz = - z^{1-\nu} C_{\nu-1} (z) = - z^{1-\nu} C'_\nu (z) - z^{-\nu} \nu C_\nu(z).$$

$$\int z^{\nu+1} C_\nu(z)\, dz = - z^{\nu+1} C_{\nu+1} (z) = \nu z^\nu C_\nu(z) - z^{\nu+1} C'_\nu (z).$$

C. Limit Considerations for $k \to 0$

The previous results can be applied for cylindrical tanks with circular cross section by letting $k \to 0$. The zeros of the determinant

$(\nu = \frac{m}{2\alpha})$

$$\Delta_\nu (\xi) = \begin{vmatrix} J'_\nu (\xi) & Y'_\nu (\xi) \\ J'_\nu (k\xi) & Y'_\nu (k\xi) \end{vmatrix} = 0$$

approach that for $k \to 0$ the value ϵ_{mn} for $J'_{\frac{m}{2\alpha}} = 0$. This is due to the fact that

$$J_\nu (x) \approx \frac{x^\nu}{2^\nu \Gamma(\nu + 1)}$$

for small x and

$$Y_\nu (x) \approx \frac{-2^\nu \Gamma(\nu)}{\pi x^\nu}$$

for $\nu > 0$ and small x. Instead of the value

$$C_{\frac{m}{2\alpha}} (\xi_{mn} \frac{r}{a})$$

in a ring sector tank, the values of $J_{\frac{m}{2\alpha}} (\epsilon_{mn} \frac{r}{a})$ have to be taken for a container of circular sector cross section.

With (N) we obtain the values L_0, L_2 for the sector tank.

$$L_0^{(\frac{m}{2\alpha})}(\epsilon_{mn}) = \frac{2}{\epsilon_{mn}} \sum_{\mu=0}^{\infty} J_{2\mu+\frac{m}{2}+1}(\epsilon_{mn}) \qquad (\mathrm{Re}\ \frac{m}{2\alpha} > -1)$$

$$L_2^{(\frac{m}{2\alpha})}(\epsilon_{mn}) = \frac{\Gamma(\frac{m}{4\alpha}+\frac{3}{2})}{\epsilon_{mn}\,\Gamma(\frac{m}{4\alpha}-\frac{1}{2})} \sum_{\mu=0}^{\infty} \frac{(\frac{m}{2\alpha}+2\mu+1)\ \Gamma(\frac{m}{2\alpha}+\mu-\frac{1}{2})}{\Gamma(\frac{m}{4\alpha}+\mu+\frac{5}{2})} J_{\frac{m}{2\alpha}+2\mu+}(\epsilon_{mn})$$

The other values can be obtained in a similar way.

TABLE I

Sector Tank

Free Surface Displacement:

$$\bar{z} = \frac{\Omega^2 e^{i\Omega t}}{g}\left[\begin{Bmatrix} r x_o(0)\ \cos\varphi \\ r y_o(0)\ \sin\varphi \end{Bmatrix} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}(0)\, J_{\frac{m}{2\alpha}}\left(\epsilon_{mn}\frac{r}{a}\right) \cdot \cos\left(\frac{m}{2\alpha}\varphi\right)\right]$$

Fluid Force:

$$F_x = m\Omega^2 e^{i\Omega t}\left[\frac{1}{h}\int_{-h}^{o}\begin{Bmatrix} x_o(z) \\ 0 \end{Bmatrix} dz + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}\sin 2\pi\alpha}{i\,\Omega\,\pi\,a\,h\,\alpha}\,\cdot\right.$$

$$\left.\cdot\left[\frac{4\alpha^2}{m^2 - 4\alpha^2)}\, J_{\frac{m}{2\alpha}}(\epsilon_{mn}) + L_o^{(\frac{m}{2\alpha})}(\epsilon_{mn})\right] \cdot \int_{-h}^{o} A_{mn}^{(x)\,(y)}(z)\, dz\right]$$

$$F_y = m\Omega^2 e^{i\Omega t}\left[\frac{1}{h}\int_{-h}^{o}\begin{Bmatrix} o \\ y_o(z) \end{Bmatrix} dz + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{[(-1)^m\cos 2\pi\alpha - 1]}{i\,\Omega\,\pi\,\alpha\,a}\cdot\frac{1}{h}\int_{-h}^{o} A_{mn}^{(x)\,(y)}(z)\, dz.\right.$$

$$\left.\cdot\left[\frac{4\alpha^2}{(m^2 - 4\alpha^2)}\, J_{\frac{m}{2\alpha}}(\epsilon_{mn}) + L_o^{\frac{m}{2\alpha}}(\epsilon_{mn})\right]\right]$$

Fluid Moment:

$$M_y = m\Omega^2 e^{i\Omega t}\left[\begin{Bmatrix} -\dfrac{1}{h}\displaystyle\int_{-h}^{o}\left(\frac{h}{2}+z\right) x_o(z)\, dz + \dfrac{x_o(-h)a^2}{4h}\left(1 + \dfrac{\sin 2\pi\alpha\cos 2\pi\alpha}{2\pi\alpha}\right) \\ 0 \qquad + \dfrac{y_o(-h)a^2}{4h}\cdot\dfrac{\sin^2 2\pi\alpha}{2\pi\alpha} \end{Bmatrix}\right. +$$

$$+ \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} \sin 2\pi\alpha}{i \Omega \pi \alpha a h} \left(\frac{4\alpha^2 a^2 L_2^{(\frac{m}{2\alpha})}(\epsilon_{mn})}{(m^2 - 4\alpha^2)} A_{mn}^{\binom{(x)}{(y)}}(-h) + \right.$$

$$\left. \left. + \left[\frac{4\alpha^2}{(m^2 - 4\alpha^2)} J_{\frac{m}{2\alpha}}(\epsilon_{mn}) + L_o^{(\frac{m}{2\alpha})}(\epsilon_{mn}) \right] \cdot \int_{-h}^{o} (\frac{h}{2} + z) A_{mn}^{\binom{(x)}{(y)}}(z)\, dz \right) \right] + mg \frac{a}{3} \frac{\sin 2\pi\alpha}{\pi\alpha}$$

$$M_x = -m\Omega^2 e^{i\Omega t} \left[\left\{ \begin{matrix} \frac{x_o(-h)a^2}{4h} \frac{\sin 2\pi\alpha}{2\pi\alpha} \\ \frac{y_o(-h)a^2}{4h} (1 - \frac{\sin 2\pi\alpha \cos 2\pi\alpha}{2\pi\alpha}) \end{matrix} + \frac{1}{h} \int_{-h}^{o} (\frac{h}{2} + z) y_o(z)\, dz \right\} + \right.$$

$$+ \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{[(-1)^m \cos 2\pi\alpha - 1]}{i \Omega \pi \alpha a h} \cdot \left[\frac{4\alpha^2 a^2}{(m^2 - 4\alpha^2)} A_{mn}^{\binom{(x)}{(y)}}(-h) L_2^{(\frac{m}{2\alpha})}(\epsilon_{mn}) + \right.$$

$$\left. \left. + \left(\frac{4\alpha^2}{(m^2 - 4\alpha^2)} J_{\frac{m}{2\alpha}}(\epsilon_{mn}) + L_o^{(\frac{m}{2\alpha})}(\epsilon_{mn}) \right) \cdot \int_{-h}^{o} (\frac{h}{2} + z) A_{mn}^{\binom{(x)}{(y)}}(z)\, dz \right] \right] -$$

$$- mg \frac{a}{3} \frac{[1 - \cos 2\pi\alpha]}{\pi\alpha}$$

TABLE II

Circular Cylinder Tank

Free Fluid Surface Displacement:

$$\bar{z} = \frac{\Omega^2 a \cos\varphi\, e^{i\Omega t}}{g} \left\{ \frac{r}{a} x_0(0) + 2 \sum_{n=1}^{\infty} \frac{J_1(\epsilon_n \frac{r}{a})\eta^2}{(\epsilon_n^2 - 1) J_1(\epsilon_n)(\frac{\epsilon_n}{a}) \cosh(\epsilon_n \frac{h}{a})(1-\eta^2)} \cdot \right.$$

$$\cdot \left[\frac{\epsilon_n g}{a\Omega^2} \left(x_0'(-h) + \int_{-h}^{0} x_0''(\zeta) \cosh\left[\frac{\epsilon_n}{a}(\zeta + h)\right] d\zeta \right) + \frac{\epsilon_n}{a} \cosh\left(\frac{\epsilon_n h}{a}\right) \cdot \right.$$

$$\left. \left. \cdot \left[x_0(0) - \frac{g\, x_0'(0)}{\Omega^2} \right] \right] \right\}$$

Fluid Force:

$$F_x = m\Omega^2 e^{i\Omega t} \left\{ \frac{1}{h} \int_{-h}^{0} x_0(z)\, dz + 2 \sum_{n=1}^{\infty} \frac{\eta^2}{(\epsilon_n^2 - 1)(\epsilon_n \frac{h}{a}) \cosh(\epsilon_n \frac{h}{a})(1-\eta^2)} \cdot \right.$$

$$\cdot \left[x_0'(-h) \left(\frac{a}{\epsilon_n} \left[1 - \cosh\left(\frac{\epsilon_n h}{a}\right) \right] + \frac{g}{\Omega^2} \sinh\left(\frac{\epsilon_n h}{a}\right) \right) + \right.$$

$$+ \int_{-h}^{0} \left(\sinh\left(\frac{\epsilon_n z}{a}\right) + \frac{\epsilon_n g}{a\Omega^2} \cosh\left(\frac{\epsilon_n z}{a}\right) \right) . \int_{-h}^{z} x_0''(\zeta) \cosh\left[\frac{\epsilon_n}{a}(\zeta + h)\right] d\zeta\, dz \Big] +$$

$$+ \left[x_0(0) - \frac{g}{\Omega^2} x_0'(0) \right] \sinh\left(\frac{\epsilon_n h}{a}\right) + \int_{-h}^{0} \cosh\left[\frac{\epsilon_n}{a}(z + h)\right] \cdot$$

$$\cdot \int_z^0 x_o''(\zeta)\left[\sinh\left(\frac{\epsilon_n}{a}\zeta\right) + \frac{\epsilon_n g}{a\Omega^2}\cosh\left(\frac{\epsilon_n \zeta}{a}\right)\right]d\zeta\, dz\Bigg]\Bigg\}$$

$$F_y = 0$$

Fluid Moment:

$$M_y = ma^2\Omega^2 e^{i\Omega t}\left\{\frac{x_o(-h)}{4h} + \frac{1}{a^2}\int_{-h}^{0}\left(\frac{1}{2}+\frac{z}{h}\right)x_o(z)dz + \right.$$

$$+ 2\sum_{n=1}^{\infty}\frac{\eta^2}{(\epsilon_n^2-1)\left(\epsilon_n \frac{h}{a}\right)\cosh\left(\epsilon_n \frac{h}{a}\right)(1-\eta^2)a^2} \cdot$$

$$\cdot\left[x_o'(-h)\left\{\left(\frac{ha}{2\epsilon_n} - \frac{ag}{\epsilon_n\Omega^2}\right) + \cosh\left(\epsilon_n \frac{h}{a}\right)\left(\frac{ah}{2\epsilon_n} + \frac{ag}{\epsilon_n\Omega^2}\right) -\right.\right.$$

$$\left. - \sinh\left(\epsilon_n \frac{h}{a}\right)\left(\frac{2a^2}{\epsilon_n^2} + \frac{gh}{2\Omega^2}\right)\right\} + \frac{a^2}{\epsilon_n^2}\int_{-h}^{0}x_o''(\zeta)\sinh\left(\frac{\epsilon_n}{a}\zeta\right)d\zeta +$$

$$+ \frac{ag}{\epsilon_n}\int_{-h}^{0}x_o''(\zeta)\cosh\left(\frac{\epsilon_n}{a}\zeta\right)d\zeta + \frac{h}{2}\int_{-h}^{0}\left[\sinh\left(\frac{\epsilon_n}{a}z\right)\cdot\right.$$

$$\cdot \int_{-h}^{z}x_o''(\zeta)\cosh\left[\frac{\epsilon_n}{a}(\zeta+h)\right]d\zeta\Bigg]dz + \frac{\epsilon_n gh}{2a\Omega^2}\int_{-h}^{0}\left[\cosh\left(\frac{\epsilon_n}{a}z\right)\cdot\right.$$

$$\cdot \int_{-h}^{z}x_o''(\zeta)\cosh\left[\frac{\epsilon_n}{a}(\zeta+h)\right]d\zeta + \frac{h}{2}\int_{-h}^{0}\left[\cosh\left[\frac{\epsilon_n}{a}(z+h)\right]\cdot\right.$$

$$\cdot \int_z^0 x_o''(\zeta)\ \sinh\left(\frac{\epsilon_n}{a}\ \zeta\right)d\zeta\Big]dz + \frac{\epsilon_n hg}{2a\Omega^2}\int_{-h}^0 \Big[\cosh\left[\frac{\epsilon_n}{a}\ (z+h)\right]\Big] \ .$$

$$\cdot \int_z^0 x_o''(\zeta)\ \cosh\left(\frac{\epsilon_n}{a}\ \zeta\right)d\zeta\Big]dz + \int_{-h}^0 \Big[z\ \sinh\left(\frac{\epsilon_n}{a}\ z\right).$$

$$\cdot \int_{-h}^z x_o''(\zeta)\ \cosh\left[\frac{\epsilon_n}{a}\ (\zeta+h)\right]d\zeta\Big]dz + \frac{\epsilon_n g}{a\Omega^2}\int_{-h}^0 \Big[z\ \cosh\left(\frac{\epsilon_n}{a}\ z\right) \ .$$

$$\cdot \int_{-h}^z x_o''(\zeta)\ \cosh\left[\frac{\epsilon_n}{a}\ (\zeta+h)\right]d\zeta\Big]dz + \int_{-h}^0 \Big[z\ \cosh\left[\frac{\epsilon_n}{a}\ (z+h)\right]\Big].$$

$$\cdot \int_z^0 x_o''\ (\zeta)\ \sinh\left(\frac{\epsilon_n}{a}\ \zeta\right)d\zeta\Big]dz + \frac{\epsilon_n g}{a\Omega^2}\quad \Big[z\ \cosh\left[\frac{\epsilon_n}{a}\ (z+h)\right]\Big] \ .$$

$$\cdot \int_z^0 x_o''\ (\zeta)\ \cosh\left(\frac{\epsilon_n}{a}\ \zeta\right)d\zeta\Big]dz - \left(x_o(0) - \frac{gx_o'(0)}{\Omega^2}\right) \ .$$

$$\cdot \left(\frac{\cosh\left(\frac{\epsilon_n}{a}\ h\right)}{\frac{\epsilon_n}{a}} - \frac{2a}{\epsilon_n} - \frac{h}{2}\ \sinh\left(\frac{\epsilon_n}{a}\ h\right)\right)\Big] \ .$$

$M_x = 0.$

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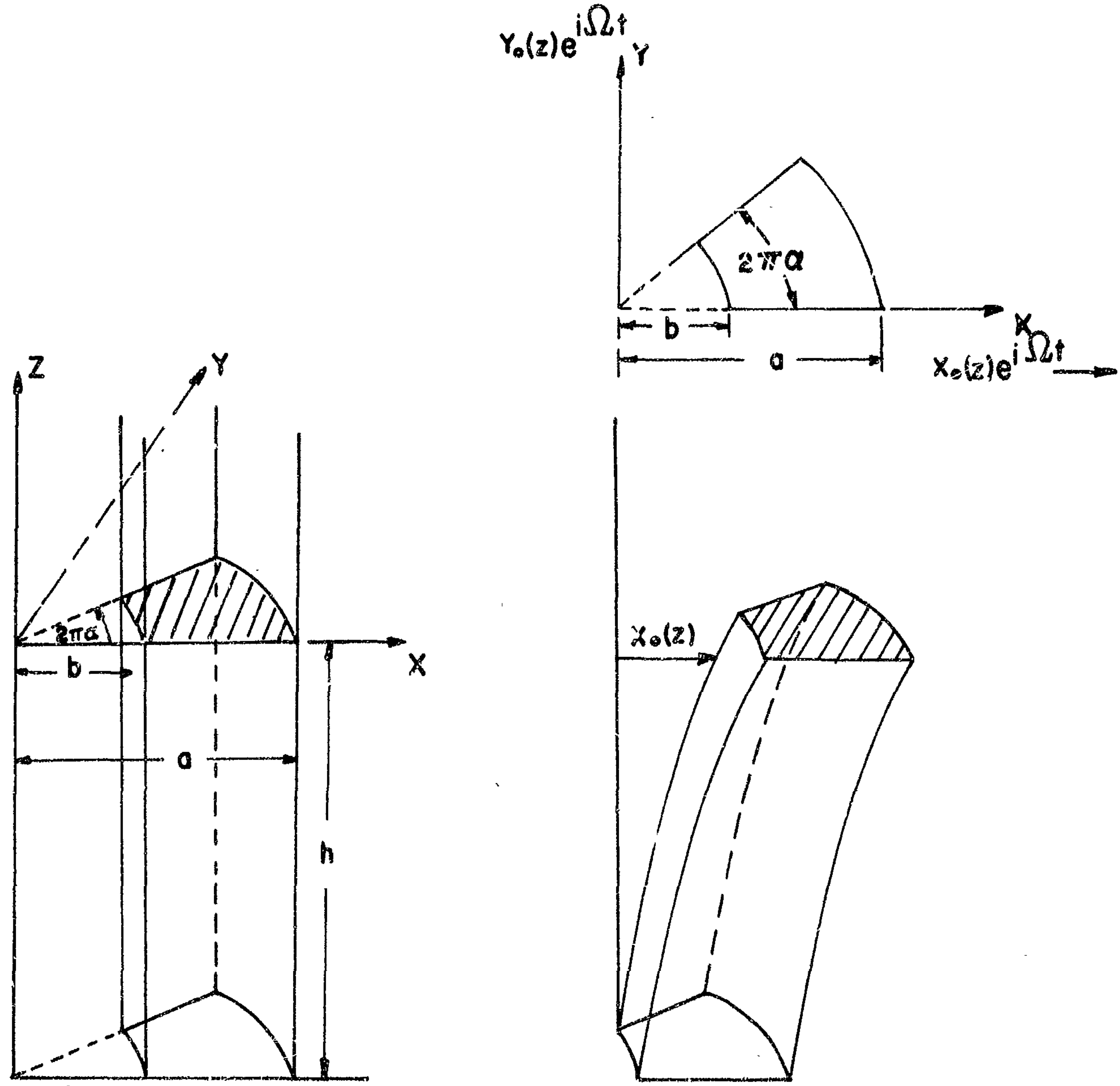

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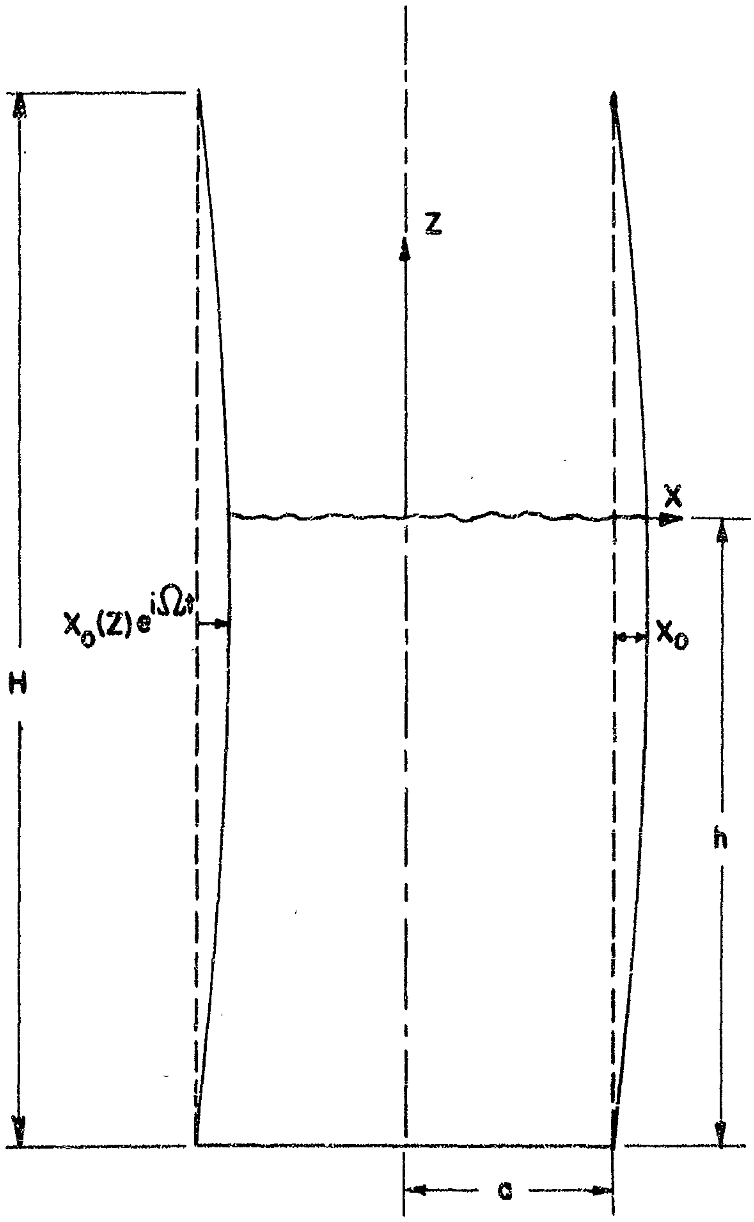
FIG. 1: TANK FORM, COORDINATE SYSTEM AND BENDING EXCITATION

$$\frac{H}{a} = 5.0$$

$$\frac{h}{a} = 3.0$$

$$X_0 = \frac{a}{10}$$



$$X_0(Z) = -\frac{4a}{250}\left[\left(\frac{Z}{a}\right)^2 + \frac{Z}{a} - 6\right]$$

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FIG. 2 BENDING TANK WALLS

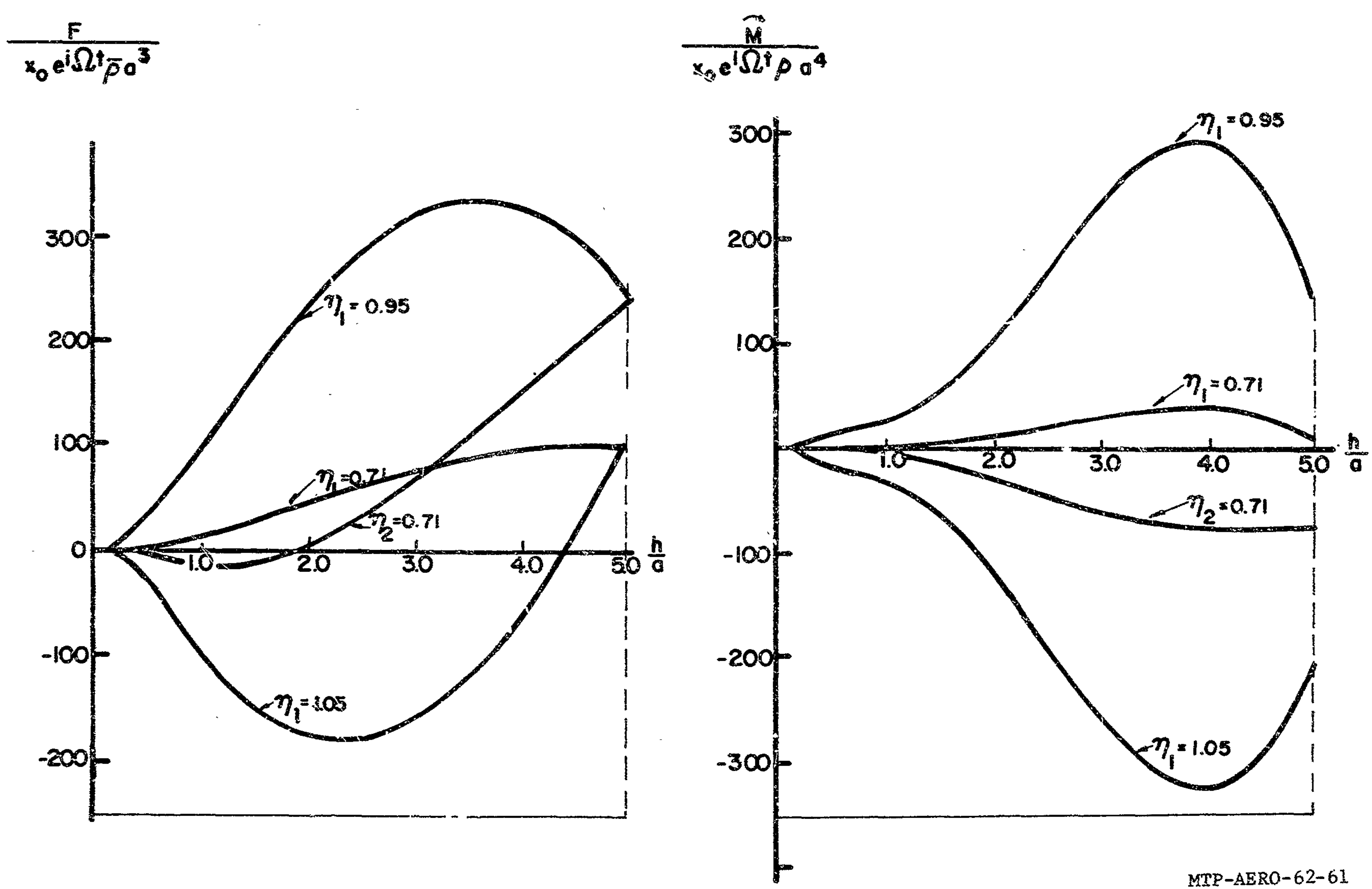

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FIG. 3 FLUID FORCE AND FLUID MOMENT DUE TO BENDING TANK WALLS FOR CIRCULAR CYLINDRICAL TANK

APPROVAL

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

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